Iteration Learning Control for Uncertain Nonlinear Systems with Time Varying Output Constraint

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OUTLINE

Part 01 Problem Formulation

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Part 04 Simulation Result

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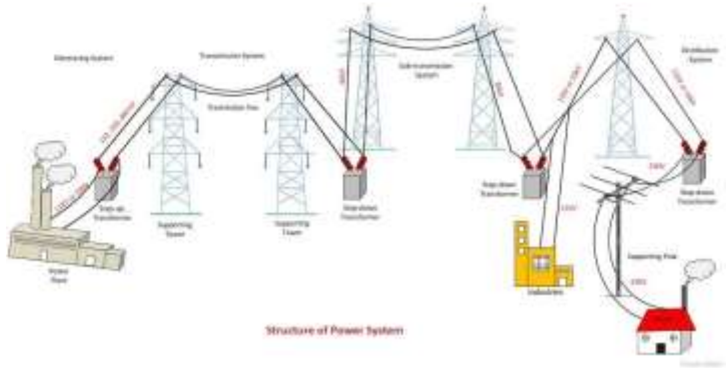
PROBLEM FORMULATION

Problem Formulation

Motivation



Power system

repetitive system

High-speed railway system

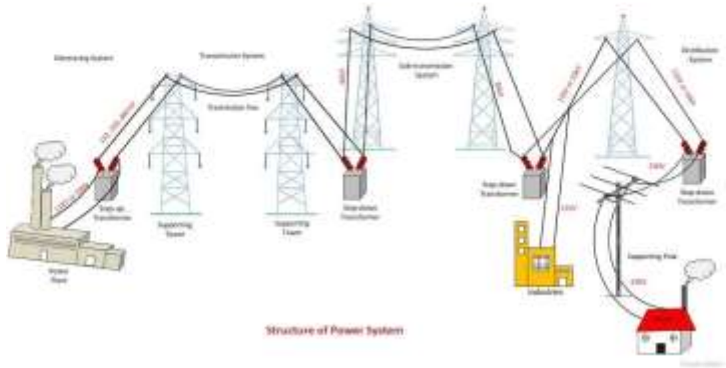
Aircraft

safety-critical

Manipulator

Motivation

How to utilize the repetitive operation pattern and guarantee the safety-critical requirement?



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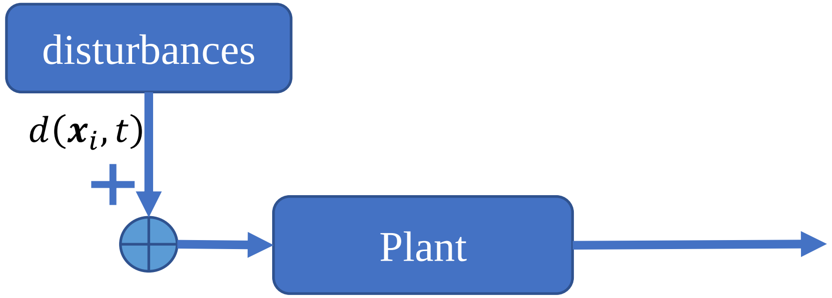
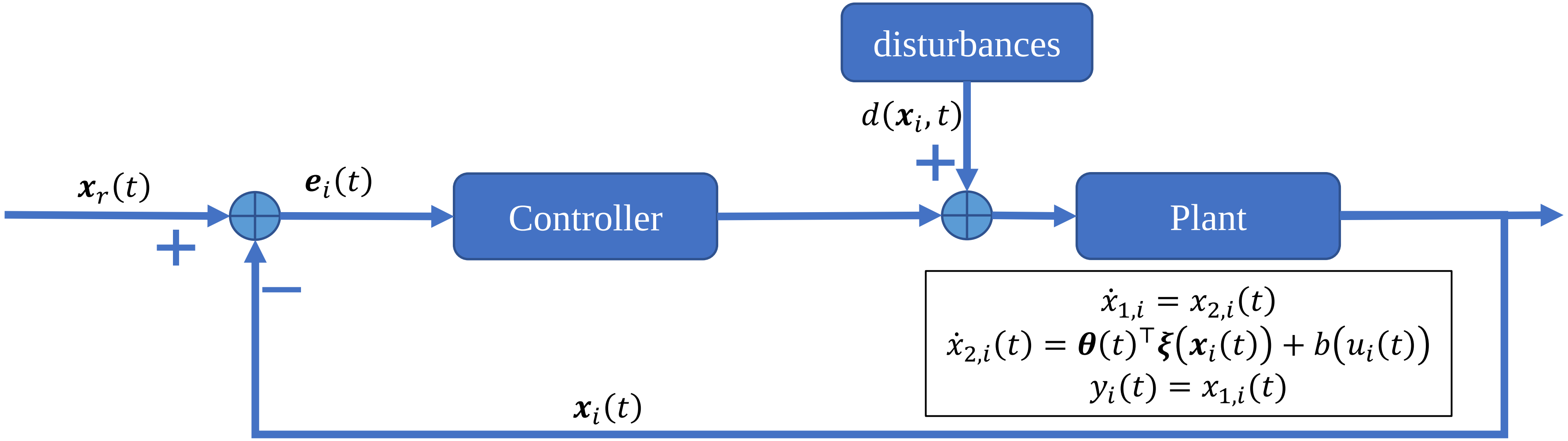


$$\dot{x}_{1,i} = x_{2,i}(t)$$
$$\dot{x}_{2,i}(t) = \boldsymbol{\theta}(t)^{\top}\boldsymbol{\xi}\big(\boldsymbol{x}_i(t)\big) + b\big(u_i(t)\big)$$
$$y_i(t) = x_{1,i}(t)$$



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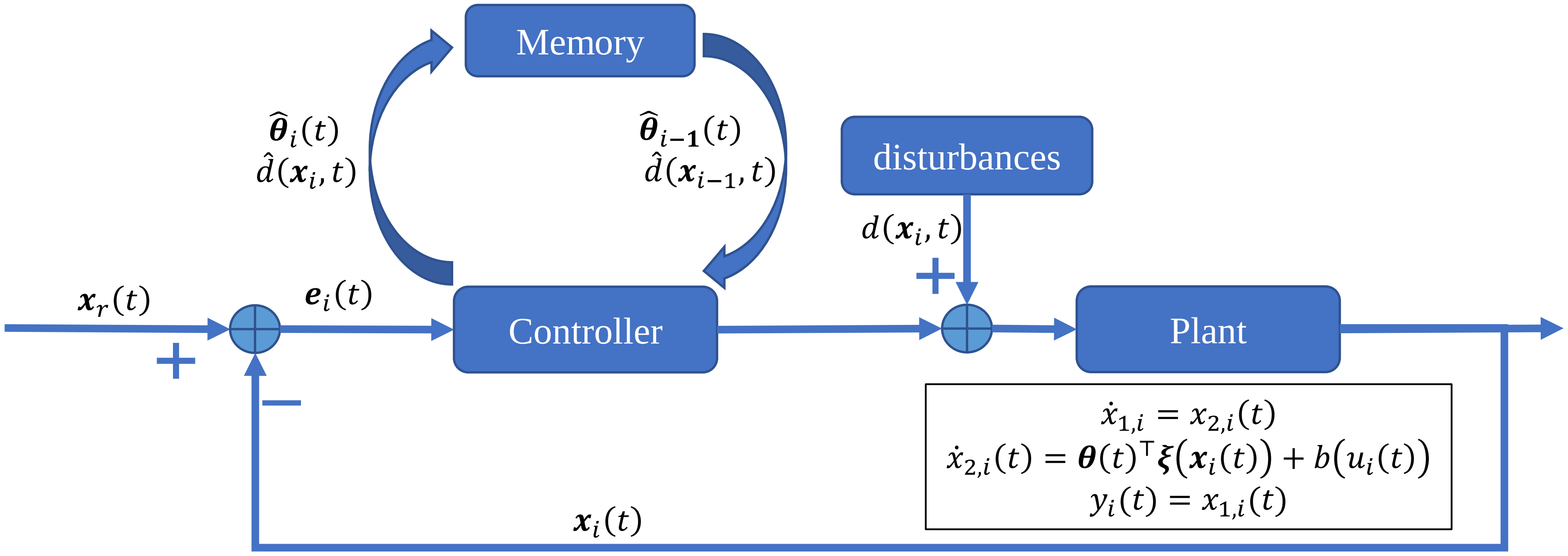

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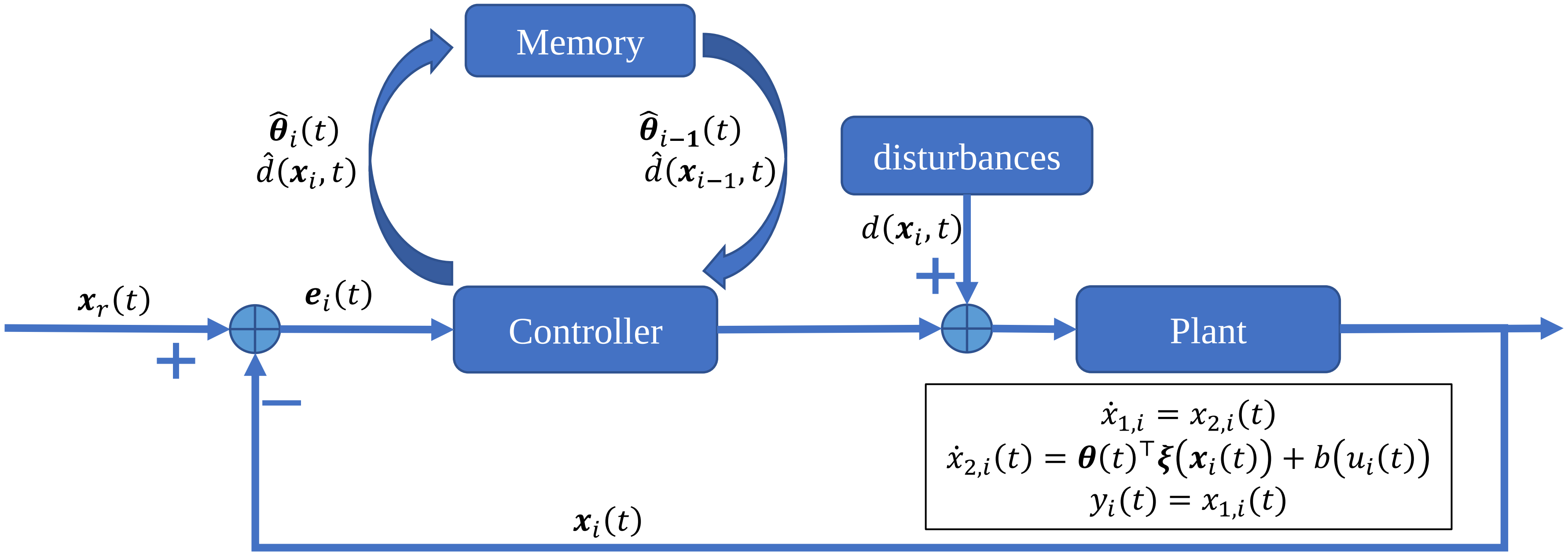

Control Objective:

Find $\{u_i\}_{i\in N}$ such that $x_i(t)$ converge to $x_r(t)$ without violating the constraints $|y_i(t)| < k_b(t)$, i.e.

$$\lim_{i\to\infty} x_i(t) = x_r(t)$$

Problem Formulation

Plant

$$\dot{x}_{1,i} = x_{2,i}(t)$$
$$\dot{x}_{2,i}(t) = \boldsymbol{\theta}(t)^{\top}\boldsymbol{\xi}\big(\boldsymbol{x}_i(t)\big) + b\big(u_i(t) + d(\boldsymbol{x}_i, t)\big)$$
$$y_i(t) = x_{1,i}(t)$$

Plant

$$\dot{x}_{1,i} = x_{2,i}(t)$$
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$$y_i(t) = x_{1,i}(t)$$

Reference model

$$\dot{x}_{1,r}(t) = x_{2,r}(t)$$
$$\dot{x}_{2,r}(t) = \boldsymbol{\theta}(t)^{\top}\boldsymbol{\xi}\big(\boldsymbol{x}_r(t)\big) + b\big(u_r(t) + d(\boldsymbol{x}_r, t)\big)$$
$$y_r(t) = x_{1,r}(t)$$

Assumptions

- (A.1) There exists $u_r(t)$ that satisfy the reference model such that $|y_r(t)| < \varepsilon_r(t)$ holds
- (A.2) The alignment condition: $x_i(0) = x_{i-1}(T)$ and $x_r(0) = x_{r-1}(T)$

$$e_i(0) = x_r(0) - x_i(0) = x_r(T) - x_{i-1}(T) = e_{i-1}(T)$$

- (A.3) $\mathrm{d_i} \triangleq d(x_i, t)$ is locally Lipschitz continuous by a known bounded function $\varphi(x_r, x_i)$

$$|d(\boldsymbol{x}_r, t) - d(\boldsymbol{x}_i, t)| \le \varphi(x_r, x_i)|x_r - x_i|$$

- (A.4) $d(x_i, t)$ and $\theta_i = \left[\theta_{1,i},\ \theta_{2,i},\ \cdots,\ \theta_{m,i}\right]^{\top}$ are bounded, i.e. $\left|\theta_{l,i}\right|_{sup} < \bar{\theta}_l,\ k = 1,2,\cdots,m$ and $|d_i|_{sup} < \bar{d}_i$

CONTROL LAW

Model-based term

Compensation term

Control law

$$u_i = \underbrace{\frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}\left[\dot{x}_{2,r}\right.}_{\text{Model-based term}} \underbrace{\left.- \widehat{\boldsymbol{\theta}}_i^\top \xi_i\right] - \widehat{d}_i + \mathrm{sign}(p(\boldsymbol{e}_i)b)\varphi_i||\boldsymbol{e}_i||}_{\text{Compensation term}}$$

Model-based term

Compensation term

Control law

$$u_i = \frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}\left[\dot{x}_{2,r} - \hat{\boldsymbol{\theta}}_i^{\top}\xi_i\right] - \hat{d}_i + \mathrm{sign}(p(\boldsymbol{e}_i)b)\varphi_i||\boldsymbol{e}_i||$$

$$V_{b,i} = k\frac{2\varepsilon_b^2}{\pi}\tan\left(\frac{\pi e_{1,i}^2}{2\varepsilon_b^2}\right)$$

tan-type BLF

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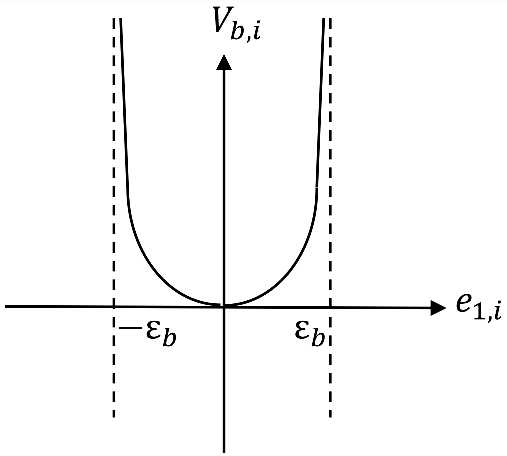

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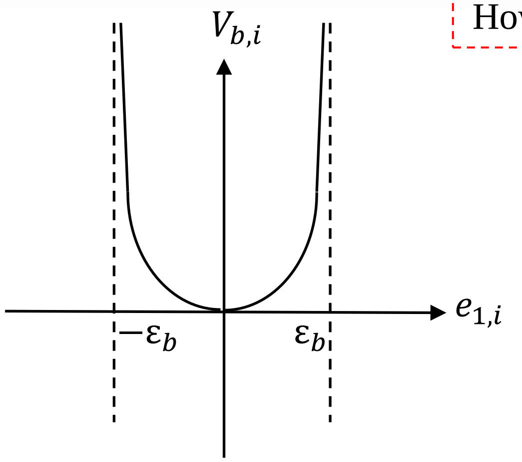

How to satisfy $|y_i(t)| < k_b(t)$?

safety-critical

A.1

$$\xrightarrow{\text{Select } \varepsilon_b(t)} 0 < \varepsilon_b(t) < k_b(t) - \varepsilon_r(t) \xrightarrow{\text{Ensure}} |e_{1,i}(t)| < \varepsilon_b(t)$$

guarantee

$$|y_i(t)| < |y_r(t)| + |e_{1,i}(t)| < \varepsilon_b(t) + \varepsilon_r(t) < k_b(t)$$

Control law

$$u_i = \boxed{\frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}[\dot{x}_{2,r}} - \hat{\boldsymbol{\theta}}_i^{\top}\xi_i] - \hat{d}_i + \mathrm{sign}(\boxed{p(\boldsymbol{e}_i)}b)\varphi_i||\boldsymbol{e}_i||$$

$$\boxed{p(e_i) = 2P_{12}e_{1,i} + 2P_{22}e_{2,i}}$$

simplified SISO system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = bv$$

control input

$$v = k\sec^2\left(\frac{\pi e_1^2}{2\varepsilon_b^2}\right)e_1 + \gamma e_2 + \frac{1}{b}\dot{x}_{2,r}$$

Control law

$$u_i = \boxed{\frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}[\dot{x}_{2,r}} - \hat{\boldsymbol{\theta}}_i^{\top}\xi_i] - \hat{d}_i + \mathrm{sign}(\boxed{p(\boldsymbol{e}_i)}b)\varphi_i||\boldsymbol{e}_i||$$

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Error dynamics:

$$\dot{e} = A_s e + B\left\{-bk\left[\sec^2\left(\frac{\pi e_1^2}{2\varepsilon_b^2}\right) - 1\right]e_1\right\}$$

$$\boxed{A_s = \begin{bmatrix} 0 & 1 \\ -bk & -b\gamma \end{bmatrix}}, B = [0\ 1]^{\top}$$

Control law

$$u_i = \frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}\left[\dot{x}_{2,r} - \widehat{\boldsymbol{\theta}}_i^{\top}\xi_i\right] - \widehat{d}_i + \mathrm{sign}(p(\boldsymbol{e}_i)b)\varphi_i||\boldsymbol{e}_i||$$

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A_s is Hurwitz

$$A_s^{\top}P + PA_s = -Q$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}$$

Symmetric positive definite matrices

Control law

$$u_i = \frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}\left[\dot{x}_{2,r} \boxed{- \hat{\boldsymbol{\theta}}_i^{\top} \xi_i\right] - \hat{d}_i + \mathrm{sign}(p(\boldsymbol{e}_i)b)\varphi_i||\boldsymbol{e}_i||}$$

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A.3 $|d_r - d_i| \le \varphi(x_r, x_i)|x_r - x_i|$

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Adaptive law:

$$\hat{\theta}_i = \mathcal{P}_\theta(\hat{\theta}_{i-1}) - \rho p(e_i)\xi_i \qquad \hat{\theta}_0 = 0$$

$$\hat{d}_i = \mathcal{P}_d(\hat{d}_{i-1}) - \beta p(e_i)b \qquad \hat{d}_0 = 0$$

Projection operations:

$$\mathcal{P}_\theta(\theta_i) = \left[\mathcal{P}_\theta(\hat{\theta}_{1,i}), \cdots, \mathcal{P}_\theta(\hat{\theta}_{m,i})\right]^{\top}$$

$$\mathcal{P}_\theta(\hat{\theta}_{l,i}) = \begin{cases} \hat{\theta}_{l,i}, & |\hat{\theta}_{l,i}| \leq \bar{\theta}_l, \\ \mathrm{sign}(\hat{\theta}_{l,i})\bar{\theta}_l, & |\hat{\theta}_{l,i}| > \bar{\theta}_l, \end{cases} \quad l = 1, \cdots, m$$

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Control law

$$u_i = \frac{\partial V_{b,i}}{\partial e_{1,i}} + \gamma e_{2,i} + \frac{1}{b}\left[\dot{x}_{2,r} - \hat{\boldsymbol{\theta}}_i^{\top}\xi_i\right] - \hat{d}_i + \operatorname{sign}(p(\boldsymbol{e}_i)b)\varphi_i||\boldsymbol{e}_i||$$

Positive gains

A.3 $|d_r - d_i| \le \varphi(x_r, x_i)|x_r - x_i|$

Adaptive law:

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SKETCH OF STABILITY ANALYSIS

Theorem

For system that meets Assumptions A.1-A.4, the control law and the adaptive learning laws guarantee that

1. $\lim_{i \to \infty} ||\boldsymbol{e}_i(t)|| = 0,\ \forall t \in [0, T]$
2. $|e_{1,i}(0)| < \varepsilon_b(0)$, then $|e_{1,i}(t)| < \varepsilon_b(t)\ \forall t \in [0, T], i = 1,2,\cdots$

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Error dynamic

$$\dot{e}_i = \mathrm{g}(\mathrm{e_i}) + \mathrm{f}(\mathrm{e_i})$$

$$g(e_i) = \left[e_{2,i}, -bk\sec\left(\frac{\pi e_{1,i}^2}{2\varepsilon_b^2}\right)e_{1,i} - b\gamma e_{2,i}\right]^\top, f(e_i) = \left[0, -\left(\theta - \hat{\theta}_i\right)^\top \xi - b\left(d_i - \hat{d}_i\right) - \mathrm{sign}(p(e_i)b)b\varphi_i||e_i||\right]^\top$$

Sketch of Stability Analysis

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non-negative barrier composite energy function (BCEF)

$$E_i(t) = V_{1,i} + V_{2,i} + V_{3,i}$$

$$V_{1,i} = k\frac{2\varepsilon_b^2}{\pi}\tan\left(\frac{\pi e_{1,i}^2}{2\varepsilon_b^2}\right);\quad V_{2,i} = \frac{1}{2\rho}\int_0^T \left(\theta - \hat{\theta}_i\right)^\top\left(\theta - \hat{\theta}_i\right)d\tau\ ;\quad V_{3,i} = \frac{1}{2\beta}\int_0^T \left(d_i - \{\hat{d}_i\}\right)^2 d\tau$$

Sketch of Stability Analysis

Part I: Difference of BCEF

$$\Delta E_i(T) = E_i(T) - E_{i-1}(T) = \Delta V_{1,i}(T) + \Delta V_{2,i}(T) + \Delta V_{3,i}(T)$$

$$(\theta - \hat{\theta}_{i-1})^\top (\theta - \hat{\theta}_{i-1}) \geq \left(\theta - \mathcal{P}_\theta(\hat{\theta}_{i-1})\right)^\top \left(\theta - \mathcal{P}_\theta(\hat{\theta}_{i-1})\right)$$

$$(d_r - d_{i-1})^2 \geq \left(d_r - \mathcal{P}_d(\hat{d}_{i-1})\right)^2$$

Sketch of Stability Analysis

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$$(d_r - d_{i-1})^2 \geq \left(d_r - \mathcal{P}_d(\hat{d}_{i-1})\right)^2$$

$$\Delta V_{1,i}(T) \leq -\int_0^T \alpha e_i^\top e_i\, d\tau - \int_0^T p(e_i)(\theta - \hat{\theta}_i)^\top \xi d\tau - \int_0^T p(e_i)b(d_i - \hat{d}_i)d\tau - \int_0^T |p(e_i)b|\varphi_i\, |e_i| d\tau$$

$$\Delta V_{2,i}(T) \leq \int_0^T p(e_i)(\theta - \hat{\theta}_i)^\top \xi_i\, d\tau$$

$$\Delta V_{3,i}(T) \leq \int_0^T p(e_i)b(d_r - \hat{d}_i)d\,\tau$$

$$p(e_i)b(d_r - d_i) \leq |p(e_i)b||d(x_r, t) - d(x_i, t)|$$

Part I: Difference of BCEF

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$$\Delta V_{2,i}(T) \leq \int_0^T p(e_i)\left(\theta - \hat{\theta}_i\right)^{\top} \xi_i \, d\tau$$

$$\Delta V_{3,i}(T) \leq \int_0^T p(e_i) b\left(d_r - \hat{d}_i\right) d\tau$$

$$p(e_i) b(d_r - d_i) \leq |p(e_i) b| |d(x_r, t) - d(x_i, t)|$$

$$\Delta E_i(T) \leq -\int_0^T \alpha e_i^{\top} e_i \, d\tau$$

$E_i(T)$ is monotonically decreasing along iteration axis

Part II: Boundedness of $E_1(t)$ and Finiteness of $E_i(t)$

$$\dot{E}_i(t) = \dot{V}_{1,i} + \dot{V}_{2,i} + \dot{V}_{3,i}$$

Part II: Boundedness of $E_1(t)$ and Finiteness of $E_i(t)$

$$\dot{E}_i(t) = \dot{V}_{1,i} + \dot{V}_{2,i} + \dot{V}_{3,i}$$

For iteration $i = 1$

- $\dot{V}_{1,1} \leq -\alpha e_1^\top e_1 - p(e_1)(\theta + p(e_1)\xi_1)^\top \xi_1 - p(e_1)b(d_1 + p(e_1)b) - |p(e_1)b|\varphi_1||e_1||$
- $\dot{V}_{2,1} \leq \frac{1}{2\rho}|\boxed{\theta}|^2 + \frac{\rho}{2}p(e_1)^2|\xi_1|^2 + p(e_1)\xi_1\theta$
- $\dot{V}_{3,1} \leq \frac{1}{2\beta}\boxed{d_r^2} + \frac{\beta}{2}p(e_1)^2 b^2 + p(e_1)bd_r$

finite

$$\dot{E}_1(t) \leq -\alpha e_1^\top e_1 + \frac{1}{2\rho}|\theta|^2 + \frac{1}{2\beta}d_r^2 - \frac{\rho}{2}p(e_1)^2|\xi_1|^2 - \frac{\beta}{2}p(e_1)^2 b^2 < \infty$$

Sketch of Stability Analysis

Part II: Boundedness of $E_1(t)$ and Finiteness of $E_i(t)$

$$\dot{E}_i(t) = \dot{V}_{1,i} + \dot{V}_{2,i} + \dot{V}_{3,i}$$

For iteration $i = 1$

- $\dot{V}_{1,1} \leq -\alpha e_1^\top e_1 - p(e_1)(\theta + p(e_1)\xi_1)^\top \xi_1 - p(e_1)b(d_1 + p(e_1)b) - |p(e_1)b|\varphi_1||e_1||$
- $\dot{V}_{2,1} \leq \frac{1}{2\rho}\boxed{|\theta|}^2 + \frac{\rho}{2}p(e_1)^2|\xi_1|^2 + p(e_1)\xi_1\theta$
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For iteration $i > 1$

- $\dot{V}_{1,i} \leq -\alpha \boldsymbol{e}_i^\top \boldsymbol{e}_i - p(\boldsymbol{e}_i)\left(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}_i\right)^\top \boldsymbol{\xi}_i - p(e_i)b\left(d_i - \hat{d}_i\right) - |p(e_i)b|\varphi_i||e_i||$
- $\dot{V}_{2,i} = \frac{1}{2\rho}\boxed{\left[||\theta||^2 - 2\theta^\top \mathcal{P}_\theta\left(\hat{\theta}_{i-1}\right) + ||\mathcal{P}_\theta\left(\hat{\theta}_{i-1}\right)||^2\right]} + p(e_i)\left[\theta - \left(\mathcal{P}_\theta\left(\hat{\theta}_{i-1}\right) - p(e_i)\xi_i\right)\right]^\top \xi_i - \frac{\rho}{2}p(e_i)^2||\xi_i||^2$
- $\dot{V}_{3,i} = \frac{1}{2\beta}\left[\boxed{d_r^2 - 2d_r\mathcal{P}\left(\hat{d}_{i-1}\right) + \mathcal{P}\left(\hat{d}_{i-1}\right)^2} + 2\beta p(e_i)bd_r - 2\beta p(e_i)b\mathcal{P}\left(\hat{d}_{i-1}\right) + \beta^2 p(e_i)^2 b^2\right]$

Finite term C_1

Finite term C_2

$$\dot{E}_i(t) \leq \frac{C_1}{2\rho} + \frac{C_2}{2\beta} < \infty$$

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Finite term C_1

Finite term C_2

$$\dot{E}_i(t) \le \frac{C_1}{2\rho} + \frac{C_2}{2\beta} < \infty$$

$$\dot{E}_i(t) < \infty \ \forall i \in N$$

Sketch of Stability Analysis

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$$\dot{E}_i(t) < \infty$$

$t \in [0, T]$

$\left|e_{1,i}(0)\right| < \varepsilon_b(0)$

$$E_i(0) = V_{1,i}\left(\mathrm{e_i}(0)\right) < \infty$$

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the boundedness of $E_i(t)$ and $V_{1,i}(t)$

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the boundedness of $E_i(t)$ and $V_{1,i}(t)$

$$|e_{1,i}(t)| < \varepsilon_b(t)\ \forall t \in [0, T], i = 1,2, \cdots$$

Part III: Convergence of State Tracking Error and Boundedness of System Output

$$\lim_{k\to\infty} E_k(T) = E_1(T) + \sum_{i=2}^{k} \Delta E_i(T)$$

Sketch of Stability Analysis

Part III: Convergence of State Tracking Error and Boundedness of System Output

$$\lim_{k\to\infty} E_k(T) = E_1(T) + \sum_{i=2}^{k} \Delta E_i(T)$$

$$\Delta E_i(T) \leq 0$$

$$\lim_{k\to\infty} E_k(T) \leq E_1(T) - \lim_{k\to\infty} \sum_{i=2}^{k} \int_0^T \alpha \boldsymbol{e}_i^{\top} \boldsymbol{e}_i \, d\tau$$

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- positiveness of $E_k(T)$
- finiteness of $E_1(T)$

$$\lim_{k\to\infty} \sum_{i=2}^{k} \int_0^T \alpha \boldsymbol{e}_i^\top \boldsymbol{e}_i \, d\tau = 0$$

$$\lim_{i\to\infty} ||\boldsymbol{e}_i(t)|| = 0, \forall t \in [0, T]$$

SIMULATIOM RESULT

Numerical Example

- Plant parameters

$$x_0(0) \triangleq [x_{1,0}(0), x_{2,0}(0)]^\top = \left[-\frac{\pi}{3}, 0\right]^\top$$

$$x_r \triangleq [x_{1,r}, x_{2,r}]^\top = \left[\sin\left(\frac{\pi}{2}t\right), \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\right]^\top, t \in [0,12]$$

$$b = 0.897$$

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$$\mathrm{b} = 0.897$$

- Parametric uncertainty

$$\theta(t) = \left[1.2\mathrm{sign}\left(\sin\left(\frac{2\pi}{3}t\right)\right), 1.2\mathrm{sign}\left(\sin\left(\frac{2\pi}{3}t\right)\right), 1.2\mathrm{sign}\left(\sin\left(\frac{2\pi}{3}t\right)\right)\right]^\top$$

$$\xi_i = [x_{1,i}, x_{1,i}^2, x_{1,i}^3]^\top$$

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$$\xi_i = [x_{1,i}, x_{1,i}^2, x_{1,i}^3]^\top$$

- External disturbance $d_i = 0.2\sin(x_{1,i}) + 0.2\sin(20\pi t)$

Numerical Example Result

Compare Case 1: $\varepsilon_b(t) = 1.1e^{-1.5t} + 0.1$
Case 2: $\varepsilon_b(t) = 1.2$

$\boldsymbol{k}$	$\boldsymbol{\gamma}$	$\boldsymbol{\rho}$	β	$\bar{d}$	$\bar{\theta}_l$
2	4	0.1	0.1	2	4

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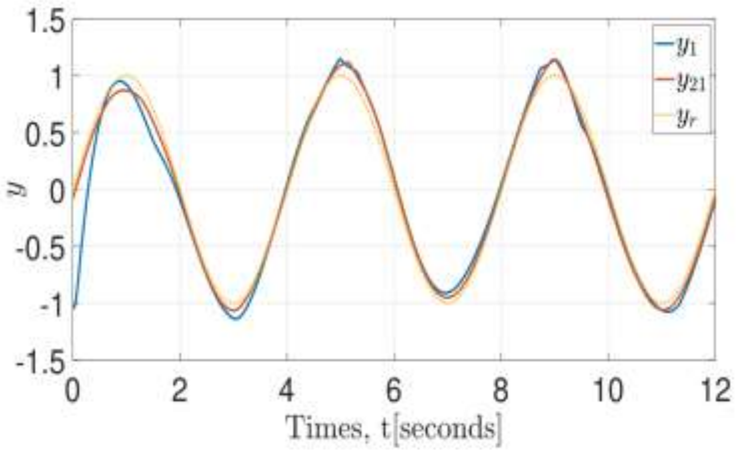

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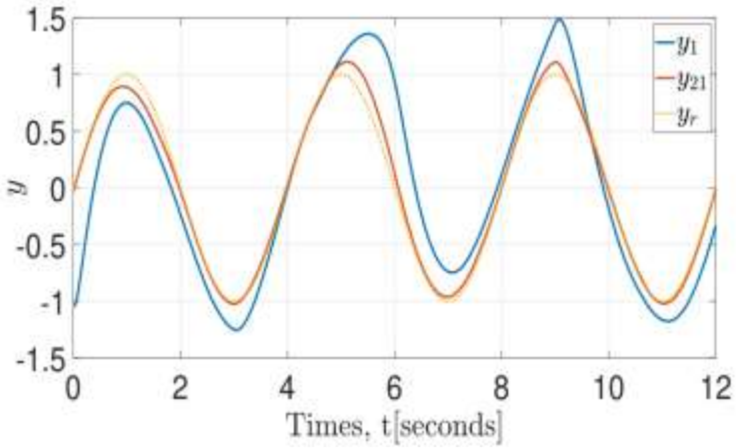

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Simulation Result

Numerical Example Result

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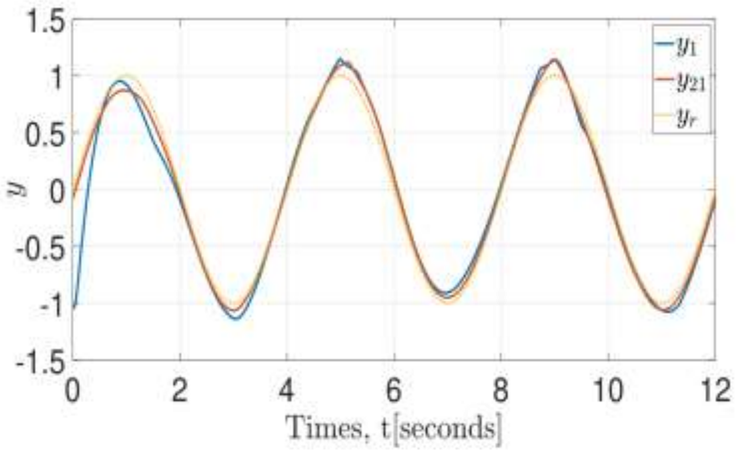

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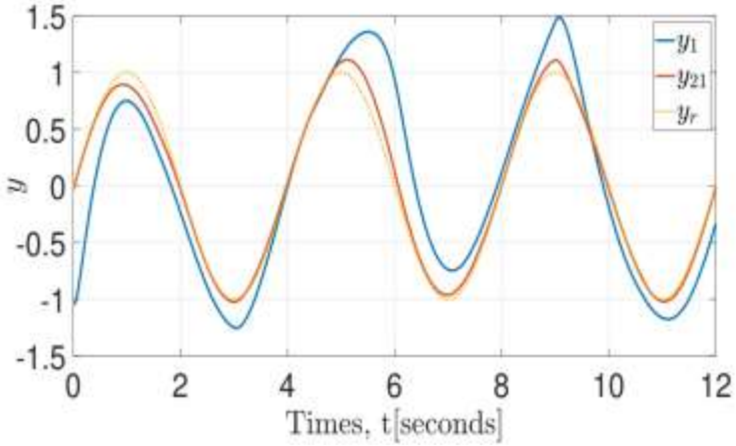

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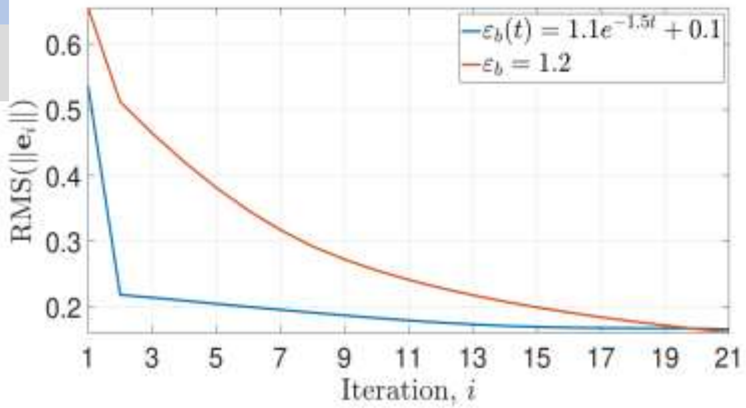
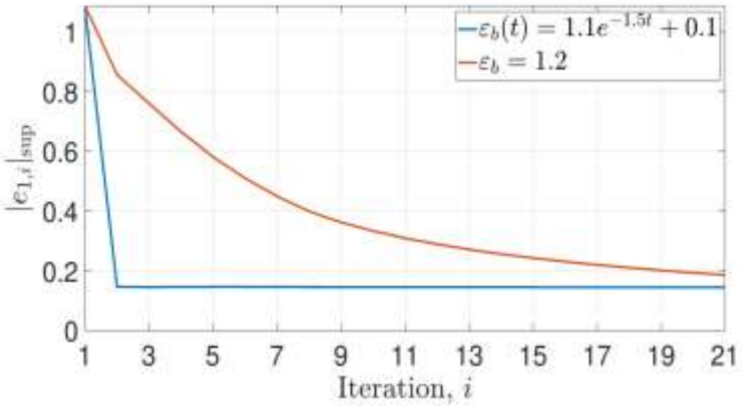
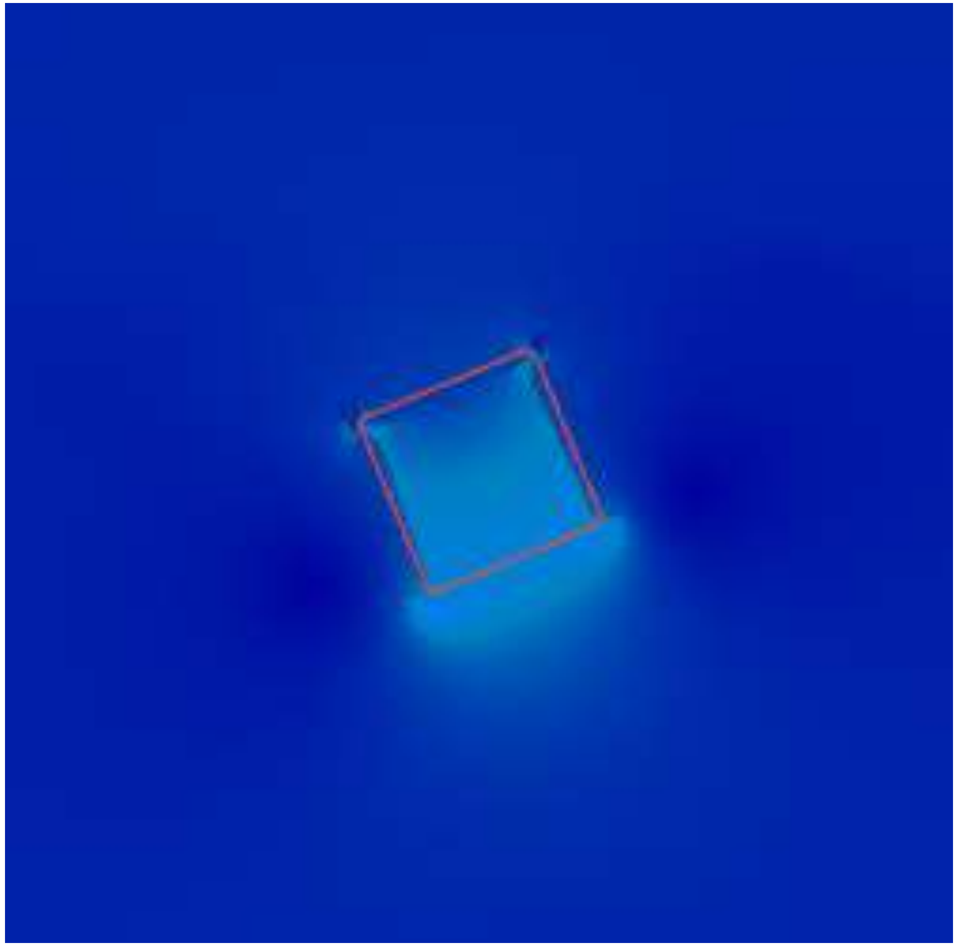




High Fidelity Simulation*

- Plant parameters $x_0(0) \triangleq \left[x_{1,0}(0), x_{2,0}(0)\right]^\top = [0.4,\ 0]^\top$

$$x_r \triangleq \left[x_{1,r}, x_{2,r}\right]^\top = \left[0.4cos\left(\frac{\pi}{2}t\right), -\frac{\pi}{5}\sin\left(\frac{\pi}{2}t\right)\right]^\top, \mathrm{t} \in [0,4]$$

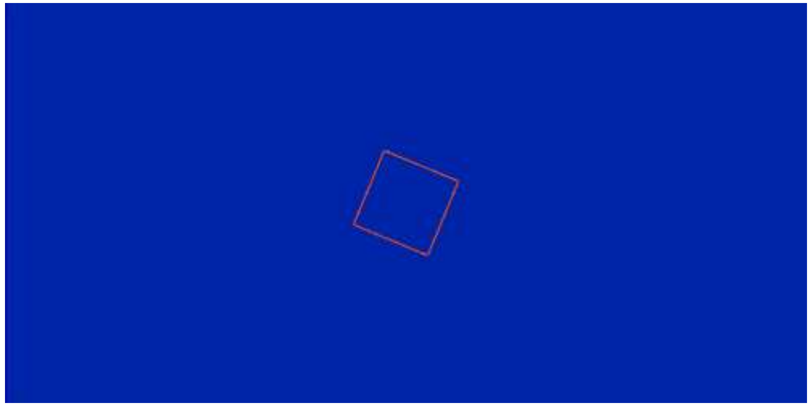


* Li, W., Chen, Y., Desbrun, M., Zheng, C., and Liu, X.(2020). Fast and scalable turbulent flow simulationwith two-way coupling. ACM Transactions on Graphics,39(4)

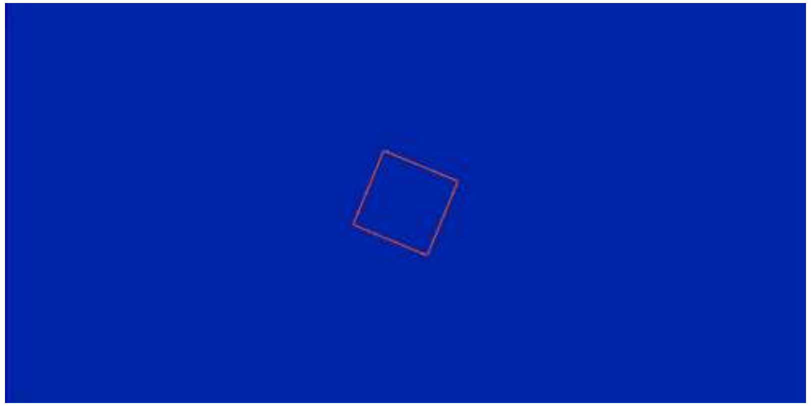
Simulation Result

High Fidelity Simulation

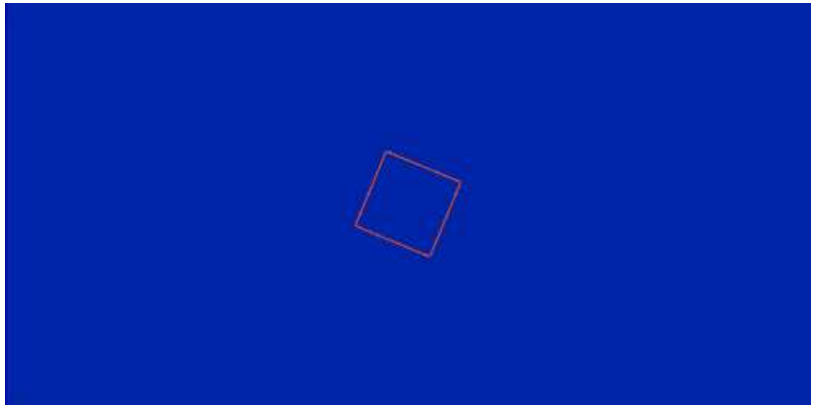
Iteration=0



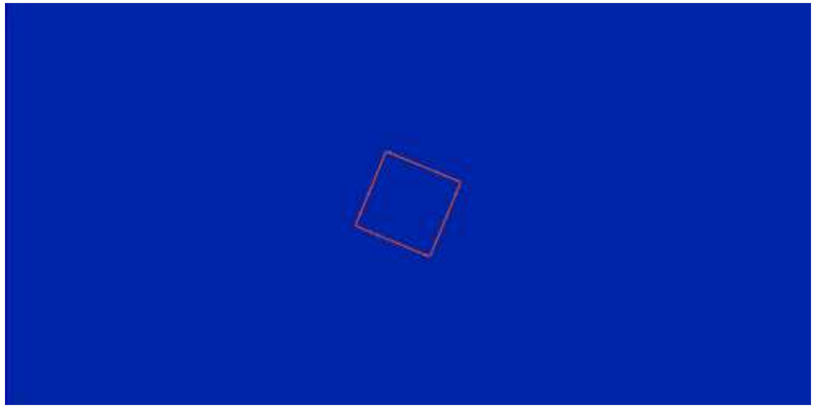
Iteration=4



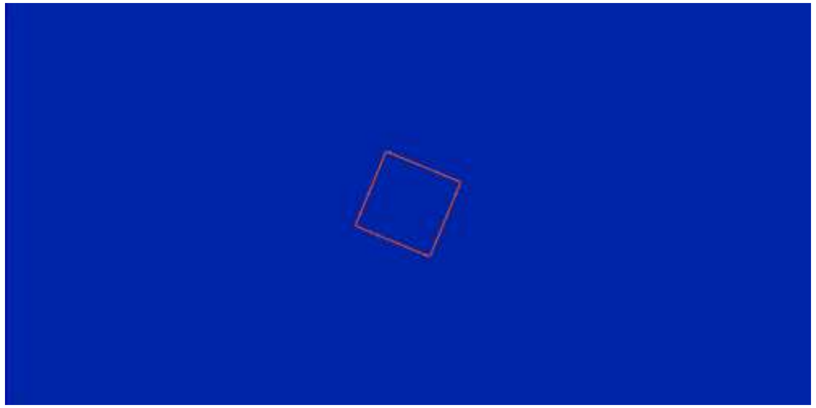
Iteration=8



Iteration=12

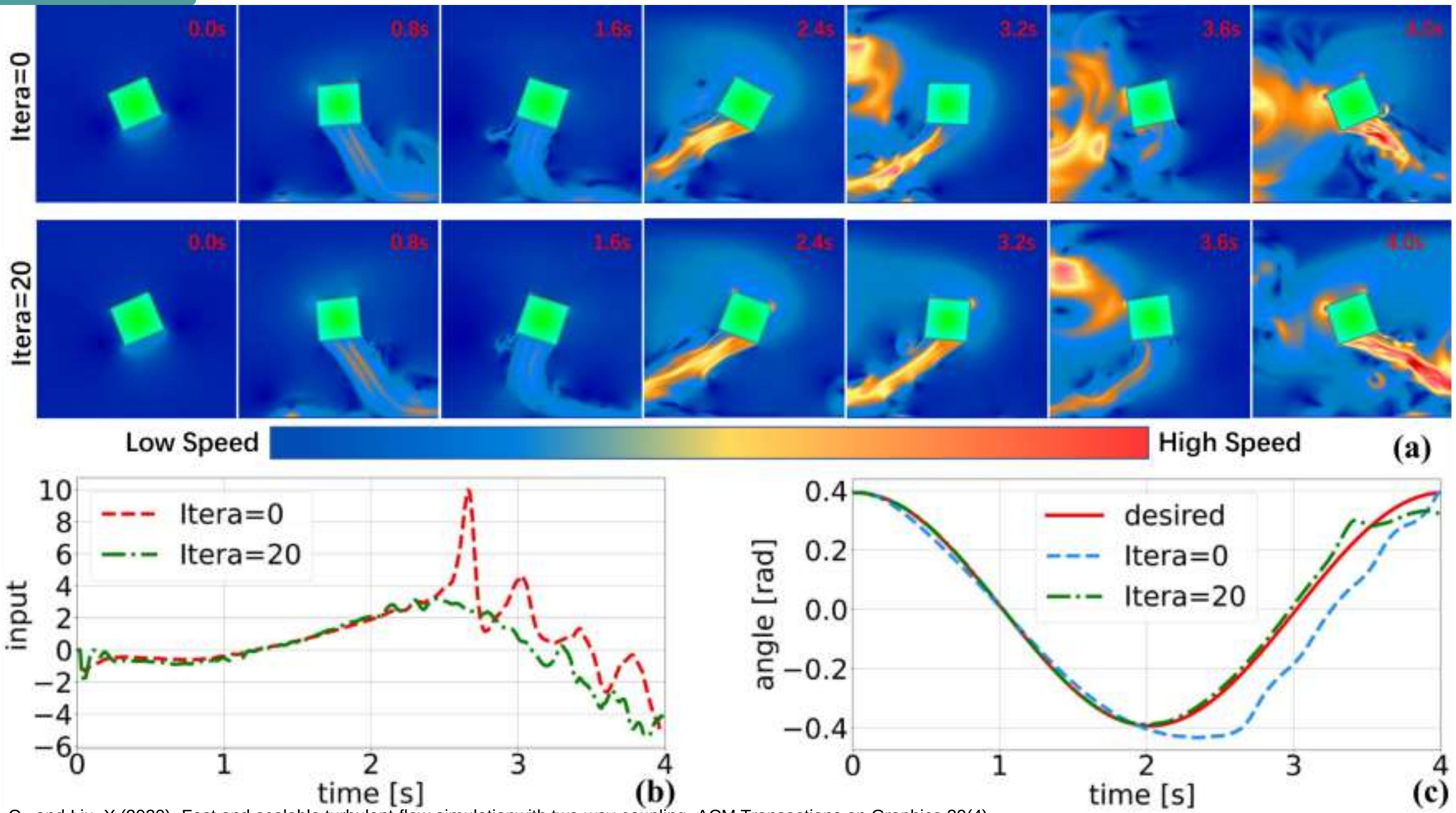


Iteration=16



Iteration=20

High Fidelity Simulation*



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CONCLUSION

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✓ The plant model has time-varying output constraint
✓ Parametric and unstructured uncertainties are both time-varying and state-dependent
✓ Guarantee the asymptotic convergence of the states to their desired values

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Drawbacks:

- Simple second-order system
- Without state constraints

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- ✓ Parametric and unstructured uncertainties are both time-varying and state-dependent
- ✓ Guarantee the asymptotic convergence of the states to their desired values

Drawbacks:

- Simple second-order system
- Without state constraints

Future Research Direction:

- ➢ Extension of high-order MIMO systems
- ➢ state constraints
- ➢ Applying to practical systems

THE END